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COMPARISON OF THE COMPOUND MATRIX AND ORTHONORMALISATION METHODS FOR CALCULATING THE STABILITY OF HEATED WATER BOUNDARY LAYERS

by

M D Thomas

Summary

A fourth-order eigenvalue finder for calculating the stability of heated water boundary layers has been developed from a sixth-order orthonormalisation method supplied by British Maritime Technology (BMT) Ltd. Comparisons have been made with a fourth-order eigenvalue finder based on the compound matrix method, also supplied by BMT, and the original sixth-order method. All the methods give similar results but there appear to be some numerical problems at high values of surface overheat.

Admiralty Research Establishment
(Teddington)
Queen's Road
Teddington, Middx TW11 OLN

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LIST OF SYMBOLS

$\bar{c}$	Dimensionless complex wavespeed (= $\omega/\alpha u_e$)
Im[ ]	Imaginary part of complex quantity
R	Displacement thickness Reynolds number (= $u_e \delta^* / \nu_\infty$)
Re[ ]	Real part of complex quantity
t	Time (s)
u	Velocity component in x-direction (m s^{-1})
$\bar{u}$	Dimensionless velocity (= u/u_e)
x	Streamwise co-ordinate (m)
y	Normal co-ordinate (m)
α	Complex wavenumber
$\bar{\alpha}$	Dimensionless complex wavenumber (= $\alpha\delta^*$)
δ^*	Boundary-layer displacement thickness
η	Transformed normal co-ordinate, defined by equation (3)
μ	Dynamic viscosity (N s m^{-2})
$\bar{\mu}$	Dimensionless viscosity (= μ/μ_∞)
ν	Kinematic viscosity ($m^2 s^{-1}$)
ρ	Density (kg m^{-3})
ϕ	Amplitude of disturbance stream function
ω	Angular frequency
$\bar{\omega}$	Dimensionless angular frequency (= $\omega\delta^*/u_e$)

Subscripts

e	Value at outer edge of boundary layer
w	Surface value
∞	Freestream value

INTRODUCTION

The calculation of the linear stability of heated water boundary layers has been tackled by previous workers using two different mathematical models. The majority of workers have assumed that the effects of heating can be modelled simply by allowing the viscosity to vary across the boundary layer, which results in a modified version of the well-known fourth-order Orr-Sommerfeld equation. Lowell & Reshotko [1] have also considered a more sophisticated model where fluctuations in all the fluid properties are included and this results in a sixth-order stability equation. BMT Ltd have developed eigenvalue solvers for both fourth- and sixth-order formulations, which have been implemented and tested at ARE by Atkins [2]. The fourth-order eigenvalue solver uses the compound matrix method and the sixth-order solver the orthonormalisation method. Some numerical experiments on flat plates with constant overheat for the temporal stability case show small differences in the resulting eigenvalues. These differences may arise solely from the different mathematical models or there may be additional effects due to the different numerical methods.

The present investigation aims to compare the compound matrix and orthonormalisation methods in solving the same stability problem and to compare eigenvalues of the fourth-order and sixth-order models using the same numerical method. To achieve this, the sixth-order stability code (using orthonormalisation) was reduced to fourth-order and compared with the existing compound matrix method. A comparison was also made between fourth-order and sixth-order eigenvalues, both obtained using the orthonormalisation method.

2. THEORY

The fourth-order linear stability equation for laminar flow over a heated body is the modified Orr-Sommerfeld equation

$$(\bar{u} - \bar{c})(\phi'' - \bar{\alpha}^2 \phi) - \bar{u}'' \phi + \frac{i}{\bar{\alpha} R} \Big\{ \bar{\mu} (\phi'''' - 2\bar{\alpha}^2 \phi'' + \bar{\alpha}^4 \phi)$$

$$+ 2\bar{\mu}' (\phi''' - \bar{\alpha}^2 \phi') + \bar{\mu}'' (\phi'' + \bar{\alpha}^2 \phi) \Big\} = 0 \qquad (1)$$

where primes represent differentiation with respect to the normal co-ordinate y/δ^*. The mean flow $\bar{u}$ is assumed to be parallel to the body surface, and small two-dimensional velocity disturbances are represented by the stream function

$$\psi(x, y, t) = \mathrm{Re}[\phi(y) \exp[i\bar{\alpha}(x - \bar{c}t)]] \qquad (2)$$

Surface overheat produces variation in the dynamic viscosity μ of the fluid, in the normal direction, which gives rise to the extra terms involving viscosity and its derivatives in equation (1). The fourth-order stability calculation finds either spatial or temporal eigenvalues of (1), the present study being limited to temporal eigenvalues $\bar{\omega}$, given the Reynolds number R (based on displacement thickness) and wavenumber $\bar{\alpha}$.

The mean flow data used in this study were calculated at 101 points on a uniform grid defined by the Howarth-Dorotnitsyn similarity variable η, viz.

$$\eta = \left\{\frac{\rho_\infty u_e}{\mu_\infty x}\right\}^{1/2} \int_0^y \frac{\rho}{\rho_\infty} dy \qquad (3)$$

Accurate velocity and temperature profiles with derivatives are available [2] for surface temperatures of 15.6°C (60°F), 32.2°C (90°F), 65.6°C (150°F) and 93.3°C (200°F), and an ambient fluid temperature of 15.6°C (60°F). Before this data could be input to a stability calculation, it was necessary to interpolate the data to more points and express it as a function of the physical co-ordinate y/δ^*. The fourth-order stability calculation requires profiles of velocity with its second derivative and viscosity with its first and second derivative, and a BMT routine known as NMIAL514 was used to calculate them. The sixth-order calculation requires profiles of velocity, temperature, viscosity, density and thermal diffusivity, with first and second derivatives, plus volumetric expansivity with first derivative and specific heat. These profiles were calculated using a modified version of a BMT routine known as BMTMF2. These routines used expressions for the fluid properties expressed as functions of temperature proposed by Lowell & Reshotko [1], including the effect of variable density, and were modified to give output at between 101 and 3201 data points.

3. NUMERICAL RESULTS

Many of the results given in the literature, e.g. those of Lowell & Reshotko [1], are for spatial eigenvalues, but the orthonormalisation methods as implemented find only temporal eigenvalues. However, Gaster [3] quotes a few very accurate temporal eigenvalues given previously by Davey for the Blasius boundary layer, i.e. the flow over an unheated flat plate. These values were used as a test example for comparison with the compound matrix method (hereafter referred to as CM4) and the fourth-order orthonormalisation method (O4). The results, presented in Table I, show that O4 gives very accurate eigen-values using input profiles specified at 801 points across the boundary layer, whereas CM4 requires 3201 points for similar accuracy. These calculations, along with most of the others presented in this Section, were carried out using double pre-cision (64 bit) arithmetic. Numerical comparisons of CM4 and O4 for a heated plate with a surface overheat of 16.6°C (30°F) are presented in Table II. Corresponding eigenvalues differ in the fifth place of decimals, and in general CM4 requires about four times as many data points as 04 to achieve the same accuracy.

In order to achieve a reliable comparison between 04 and the original sixth-order orthonormalisation method (06), the sixth-order interpolation code BMTMF2 was modified to produce profiles which could be read in directly by 04. Table III shows comparisons between 04 and 06 for a plate with overheat of 16.6°C (30°F), using 801 mean flow data points. Corresponding eigenvalues differ in the fourth decimal place. Tables IV and V show results similar to those in Tables II and III respectively for an overheat of 77.7°C (140°F). 04 and CM4 eigenvalues differ in the third or fourth decimal place and 04 and 06 eigenvalues in the third decimal place. At this high value of surface overheat, CM4 eigenvalues computed using 400 data points are significantly in error, and 06 fails to converge at Reynolds numbers at or above 10,000. It should be noted, by comparison of Tables II and III or Tables IV and V, that corresponding 04 eigenvalues differ in the fifth place of decimals. These differences arise from the use of different interpolation codes (NMIAL514 and BMTMF2) to calculate the input profiles.

The domains of convergence for 04 and CM4 were found for two different eigenvalues at a surface overheat of 16.6°C (30°F), using 801 data points. The results are presented in Figure 1. At $R=400$ and $\bar{\alpha}=0.05$, the domain of convergence for 04 is slightly larger than that for CM4, whereas at $R=20,000$ and $\bar{\alpha}=0.15$, 04 has a much larger domain of convergence than CM4. It should be noted that 04 requires two initial guesses whereas CM4 requires only one. For these tests, two values relatively close together were chosen.

4. CONCLUSIONS

Eigenvalues of temporal stability have been calculated for unheated and heated flat plate boundary layers using numerical methods based on compound matrices and orthonormalisation, and fourth- and sixth-order mathematical models. The compound matrix method requires about four times as many grid points across the boundary layer as the orthonormalisation method to achieve the same accuracy. Nontrivial but small differences in eigenvalues arise from using different numerical methods, different interpolation codes and a sixth-order rather than a fourth-order mathematical model. However these differences are of no practical significance in engineering calculations of flow transition based on the calculation of amplification factors. The fourth-order orthonormalisation method is often much less sensitive to the initial guess than the compound matrix method. At Reynolds numbers above 10,000 with large surface overheat, the sixth-order orthonormalisation method fails to converge.

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M D Thomas (CASE Student)

MDT/RJE

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	No. of Data Points	Compound Matrices Frequency, $\bar{\omega}$ Real	Imag	Orthonormalisation Frequency, $\bar{\omega}$ Real	Imag
R=500	101	0.11837108	0.00043163	0.11927733	-0.00026881
$\bar{\alpha}$=0.3	201	0.11920156	-0.00024126	0.11930217	-0.00027839
	401	0.11929573	-0.00027810	0.11930368	-0.00027985
	801	0.11930321	-0.00027989	0.11930376	-0.00027998
	1601	0.11930372	-0.00027997	0.11930377	-0.00027998
	3201	0.11930376	-0.00027998	0.11930377	-0.00027998
	Davey result	....		0.11930376	-0.00027998
R=1500	101	0.06094088	0.00681870	0.06308966	0.00315653
$\bar{\alpha}$=0.2	201	0.06272070	0.00335158	0.06312052	0.00315793
	401	0.06308739	0.00316429	0.06312277	0.00315676
	801	0.06312033	0.00315687	0.06312291	0.00315663
	1601	0.06312274	0.00315664	0.06312291	0.00315663
	3201	0.06312290	0.00315663	0.06312291	0.00315663
	Davey result	....		0.06312291	0.00315663
R=3000	101	0.03876340	0.01055162	0.04018800	0.00277420
$\bar{\alpha}$=0.15	201	0.03935751	0.00325268	0.04021674	0.00278175
	401	0.04013689	0.00279521	0.04021904	0.00278080
	801	0.04021302	0.00278075	0.04021918	0.00278068
	1601	0.04021876	0.00278069	0.04021919	0.00278068
	3201	0.04021916	0.00278071	0.04021919	0.00278067
	Davey result	....		0.04021919	0.00278068

Table I Effect of number of data points on eigenvalues for Blasius flow

	No. of Data Points	Compound Matrices Frequency, $\bar{\omega}$ Real	Compound Matrices Frequency, $\bar{\omega}$ Imag	Orthonormalisation Frequency, $\bar{\omega}$ Real	Orthonormalisation Frequency, $\bar{\omega}$ Imag
R=5000	401	0.02672564	-0.00053033	0.02687768	-0.00056928
	801	0.02688539	-0.00057322	0.02687812	-0.00056975
$\bar{\alpha}$=0.12	1601	0.02689827	-0.00057377	0.02687815	-0.00056979
	3201	0.02689917	-0.00057371	0.02687815	-0.00056979
R=6000	401	0.03583790	0.00038594	0.03605866	0.00019017
	801	0.03607127	0.00019029	0.03606103	0.00018960
$\bar{\alpha}$=0.16	1601	0.03609431	0.00017986	0.03606118	0.00018952
	3201	0.03609607	0.00017933	0.03606119	0.00018952
R=7000	401	0.02553635	0.00020172	0.02579902	0.00007982
	801	0.02579891	0.00007993	0.02579999	0.00007926
$\bar{\alpha}$=0.12233	1601	0.02582203	0.00007605	0.02580005	0.00007921
	3201	0.02582371	0.00007599	0.02580005	0.00007920
R=8950	401	0.02068249	0.00021632	0.02102659	0.00007213
	801	0.02101595	0.00007424	0.02102736	0.00007161
$\bar{\alpha}$=0.10668	1601	0.02104530	0.00007098	0.02102742	0.00007156
	3201	0.02104744	0.00007105	0.02102742	0.00007156
R=14250	401	0.03317202	0.00136906	0.03324520	0.00001425
	801	0.03322688	0.00010979	0.03325574	0.00001756
$\bar{\alpha}$=0.17315	1601	0.03328664	0.00000889	0.03325657	0.00001758
	3201	0.03329300	0.00000215	0.03325662	0.00001757
R=400	401	0.01554260	-0.00736119	0.01559615	-0.00747471
	801	0.01554284	-0.00736108	0.01559615	-0.00747471
$\bar{\alpha}$=0.05	1601	0.01554285	-0.00736108	0.01559615	-0.00747471
	3201	0.01554285	-0.00736108	0.01559615	-0.00747471
R=2000	401	0.02555008	-0.00371474	0.02558157	-0.00372007
	801	0.02557896	-0.00371171	0.02558150	-0.00372014
$\bar{\alpha}$=0.1	1601	0.02558096	-0.00371123	0.02558149	-0.00372014
	3201	0.02558109	-0.00371119	0.02558149	-0.00372014
R=10000	401	0.03003820	0.00115138	0.03043131	0.00061624
	801	0.03041710	0.00064024	0.03043521	0.00061618
$\bar{\alpha}$=0.15	1601	0.03046383	0.00061023	0.03043549	0.00061610
	3201	0.03046768	0.00060861	0.03043550	0.00061609
R=20000	401	0.02657391	0.00258528	0.02666037	0.00053342
	801	0.02659262	0.00068348	0.02667001	0.00053709
$\bar{\alpha}$=0.15	1601	0.02669105	0.00053670	0.02667079	0.00053714
	3201	0.02670150	0.00052711	0.02667085	0.00053713

Table II Effect of number of data points on heated flat-plate eigenvalues

(T_∞= 15.6°C (60°F), T_w =32.2°C (90°F))

Reynolds Number, R	Wave-number $\bar{\alpha}$	Fourth-order Frequency, $\bar{\omega}$ Real	Fourth-order Frequency, $\bar{\omega}$ Imag	Sixth-order Frequency, $\bar{\omega}$ Real	Sixth-order Frequency, $\bar{\omega}$ Imag
5000	0.12	0.02688003	-0.00057423	0.02681933	-0.00070606
6000	0.16	0.03606273	0.00018400	0.03607933	0.00013684
7000	0.12233	0.02580160	0.00007486	0.02578732	0.00000232
8950	0.10668	0.02102873	0.00006782	0.02101412	0.00000140
14250	0.17315	0.03325552	0.00001264	0.03333099	-0.00000042
400	0.05	0.01559874	-0.00747565	0.01529915	-0.00630088
2000	0.10	0.02558384	-0.00372396	0.02476187	-0.00431349
10000	0.15	0.03043617	0.00061122	0.03047583	0.00058985
20000	0.15	0.02666979	0.00053281	0.02673102	0.00052394

Table III Comparison of eigenvalues using fourth-order and sixth-order approaches

(Orthonormalisation method, T_∞= 15.6°C (60°F), T_W= 32.2°C (90°F), 801 mean flow data points)

No. of Data Points		Compound Matrices Frequency, $\bar{\omega}$ Real	Imag	Orthonormalisation Frequency, $\bar{\omega}$ Real	Imag
	401	0.01536584	-0.00996988	0.01433481	-0.00866549
R=400	801	0.01536766	-0.00996953	0.01433474	-0.00866549
$\bar{\alpha}$=0.05	1601	0.01536779	-0.00996950	0.01433474	-0.00866549
	3201	0.01536779	-0.00996950	0.01433474	-0.00866549
	401	0.02048911	-0.00514367	0.02076047	-0.00477339
R=2000	801	0.02065693	-0.00516084	0.02075991	-0.00477347
$\bar{\alpha}$=0.1	1601	0.02066983	-0.00515970	0.02075979	-0.00477349
	3201	0.02067071	-0.00515940	0.02075978	-0.00477349
	401	0.03808205	-0.01629625	0.02424646	0.00009607
R=10000	801	0.02386329	0.00015183	0.02428397	0.00011381
$\bar{\alpha}$=0.15	1601	0.02414258	0.00000271	0.02428719	0.00011292
	3201	0.02416821	-0.00000321	0.02428739	0.00011277
	401	0.04828146	-0.02063293	0.02114929	0.00062425
R=20000	801	0.02046138	0.00144013	0.02121030	0.00069691
$\bar{\alpha}$=0.15	1601	0.02102590	0.00067174	0.02121867	0.00069828
	3201	0.02110003	0.00063098	0.02121930	0.00069814

Table IV Effect of number of data points on heated flat plate eigenvalues

(T_∞= 15.6°C (60°F), T_w= 93.3°C (200°F))

Reynolds Number, R	Wave-number $\bar{\alpha}$	Fourth-order Frequency, $\bar{\omega}$ Real	Fourth-order Frequency, $\bar{\omega}$ Imag	Sixth-order Frequency, $\bar{\omega}$ Real	Sixth-order Frequency, $\bar{\omega}$ Imag
5000	0.12	0.02175446	-0.00178270	0.02190792	-0.00181386
6000	0.16	0.02879021	-0.00066548	0.02924572	-0.00068008
7000	0.12233	0.02083715	-0.00098930	0.02108551	-0.00100218
8950	0.10668	0.01706483	-0.00088341	0.01725744	-0.00088598
14250	0.17315	0.02643489	0.00049906	failed to converge	
400	0.05	0.01458126	-0.00872240	0.01314170	-0.00646626
2000	0.10	0.02117409	-0.00474457	0.02034761	-0.00457443
10000	0.15	0.02479067	-0.00017715	0.02473125	0.00007119
20000	0.15	0.02167021	0.00072690	failed to converge	

Table V Comparison of eigenvalues using fourth-order and sixth-order approaches

(Orthonormalisation method, T_∞= 15.6°C (60°F), T_w= 93.3°C (200°F), 801 mean-flow data points)

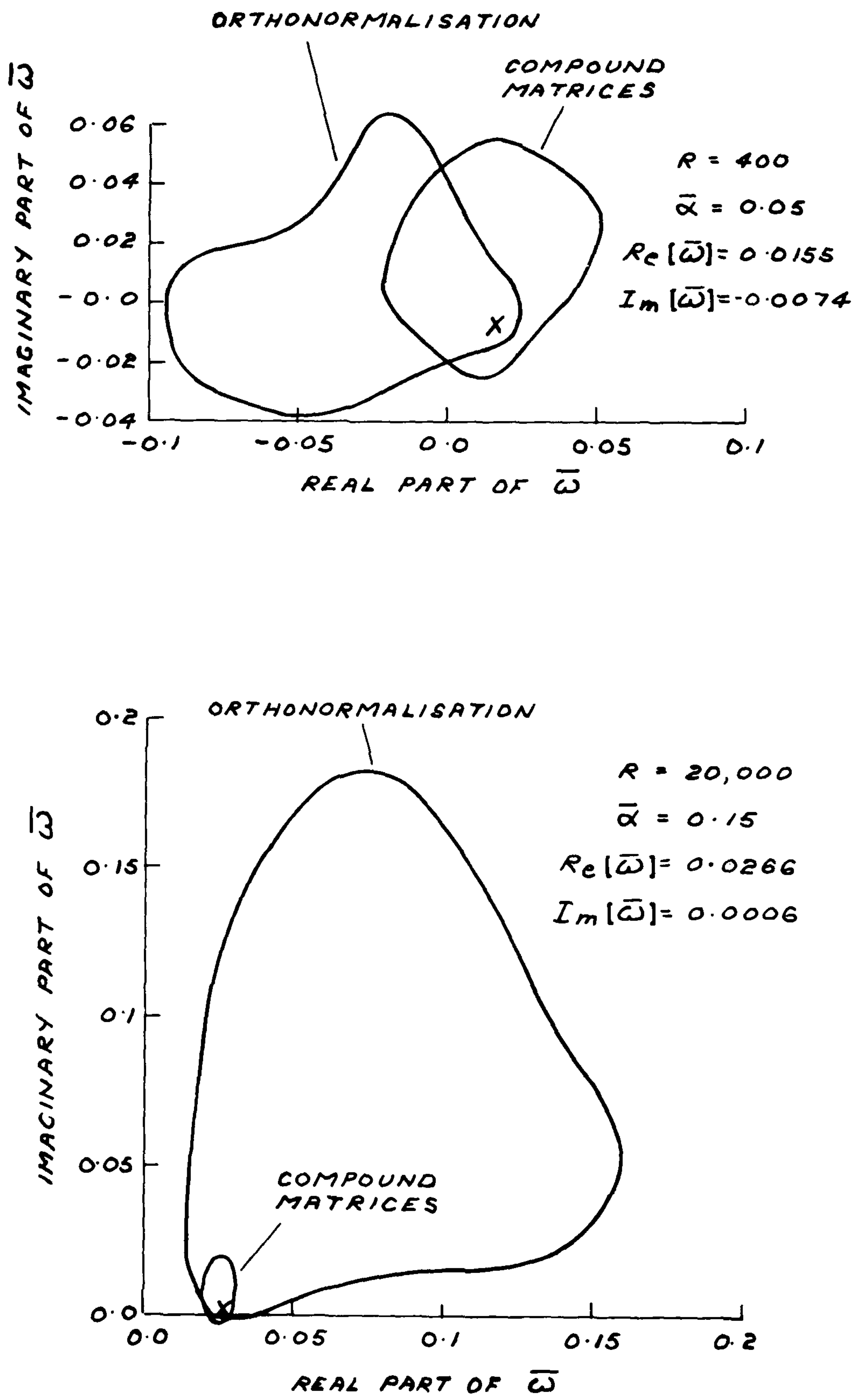

FIG.1. GRAPHS SHOWING AREA OF CONVERGENCE FOR TYPICAL EIGENVALUE CALCULATIONS

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